



Grade 6 Math Circles

Week of 17th October

Newton's Second Law

Who was Newton? What are his laws?

Sir Isaac Newton was an English mathematician, physicist, astronomer, alchemist, theologian, and author. At the time, he was called a *Natural Philosopher*. A man of many talents, one which delved into a deep interest for all things the world had to offer. One of his most significant contributions – aside from inventing calculus – was to the field of physics. Newton formulated the laws of motion and universal gravitation that paved the way for modern physics and our general understanding of the entire universe until it was superseded by the theory of relativity. Isaac Newton put forward the following laws of motion

- I. A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.
- II. When a body is acted upon by a net force, the body's acceleration multiplied by its mass is equal to the net force.
- III. If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

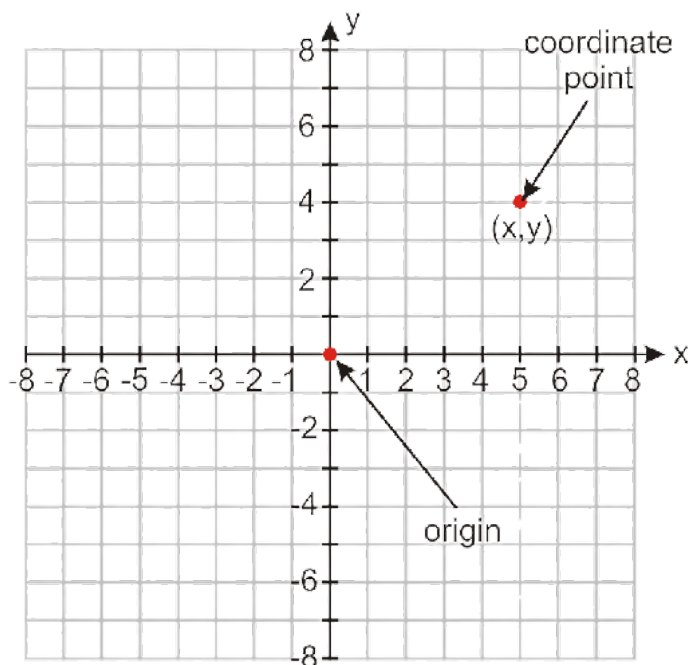
Newton was just 23 years of age when he discovered these fundamental equations! In this lesson, we'll be taking a closer look at the second law. The second law has been the basis for many fields of physics since its discovery. Since then, many different physicists have reinterpreted the law, rewritten it in their own words, in their own ways, which furthered our understanding of the universe.

Stop and Think

Although we'll dive into the second law, think about the first and third. What do you think they mean? Can you write down an equation to interpret them mathematically?

The Cartesian Plane

Many of you probably haven't seen the Cartesian plane, or if you have, don't how to work with it. The Cartesian plane is a 2-dimensional coordinate plane which is formed by the intersection of the x -axis and y -axis. Where they intersect, we call it the origin $(0,0)$. Any other point on the plane is denoted (x,y) , where x represents the horizontal coordinate, and y , the vertical component.



We use the Cartesian plane to draw 2 dimensional shapes like vectors, functions, even shapes.

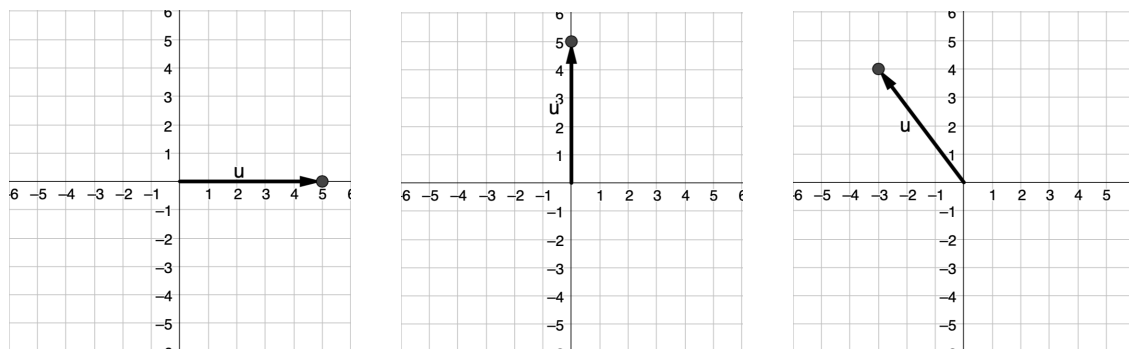
Exercise 1

On the Cartesian plane, label the following points and connect the points to the origin.

- $(1, 1)$
- $(0, 4)$
- $(-2, 4)$
- $(-\frac{1}{2}, 2)$

Vectors

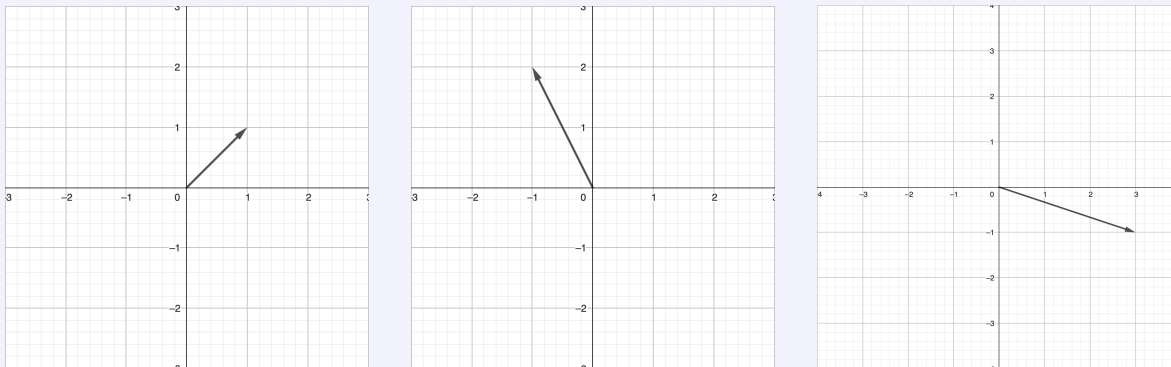
Vectors are very important in physics. In two-dimensional space they are denoted (a, b) , where a is the x -component and b is the y -component. Vectors are defined as quantities that have both a **magnitude** and a **direction**. Magnitude is just a number, how *large* the quantity is, and the direction tells us where it is pointing. We normally denote a vector as a letter with an arrow on top, like \vec{u} . Graphically, say we have a vector \vec{u} , it could look like



Starting from the left, the vector is $(5, 0)$, $(0, 5)$, and finally $(-3, 4)$. The vector \vec{u} has magnitude 5 in all examples. We'll talk about magnitude later on in the lesson – we'll come to realize that this may not be as easy as just looking at the graph, and will actually need some math. We use vectors to represent forces acting on objects, as forces come with a magnitude and a direction (the strength and direction of the force). In physics, a force can be applied to a mass in any direction, often there are multiple forces acting on a body from multiple directions at a time!

Exercise 1

Given the following vectors, write out their x and y components using vector form.

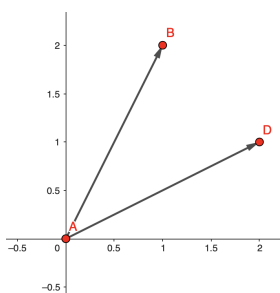


Let's quickly go over some very important terms that are associated with vectors!

- Magnitude: The magnitude of the vector represents the length of the vector.
- Direction: The direction of a vector is the direction the vector points (with respect to the origin of the coordinate system).
- Unit Vectors: A unit vector is a vector which has magnitude 1.

Vector Operations

There are some vector operations as well that we have to consider. Adding, and subtracting vectors! First, imagine I apply a force upwards (**N**), and a force to the right (**E**); if we talk about the net force, we'd want to add our forces together, or sometimes have to subtract them. Thankfully, vector addition is quite simple, tip to tail! Imagine you have two vectors you'd like to add;



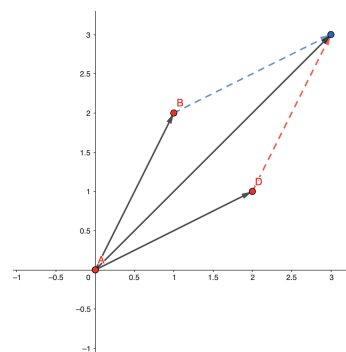
Here we have the vector \vec{v} from A to B and the vector \vec{u} from A to D . In vector form, we have that $\vec{v} = (1, 2)$ and $\vec{u} = (2, 1)$. How we define vector addition is quite simple, we *slide* the first vector so they meet up *tip to tail*. You then draw the new vector, which is the addition of the two vectors, starting from the tail of the first, and the tip of the second. You can do this with either vector, we'll use red for the vector \vec{v} , and blue for \vec{u} . Visualizing this on our graph, it looks like

So, we've created a new vector, \vec{w} , which is defined as $\vec{v} + \vec{u}$. Look carefully at the vector form of \vec{w} , it's $(3, 3)$. So, it turns out, we define vector addition component wise, that is

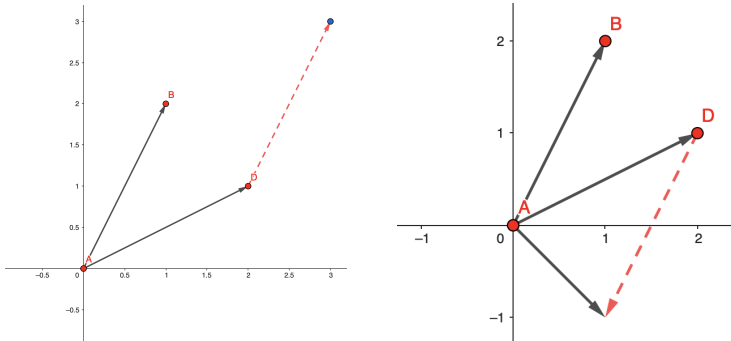
$$(a, b) + (c, d) = (a + c, b + d)$$

It also happens to be the case that vector subtraction is simply

$$(a, b) - (c, d) = (a, b) + (-c, -d) = (a - c, b - d)$$



Visually, subtracting vectors is very similar, this time, when we subtract two vectors we connect tip to tail but we flip the one vector we dragged to the tip in the opposite direction. So, using the same colours, and doing $\vec{v} - \vec{u}$, we have



Now making sure that this is true, we see that $\vec{v} - \vec{u} = (1, 2) - (2, 1) = (-1, 1)$, which is exactly what we see in the image above!

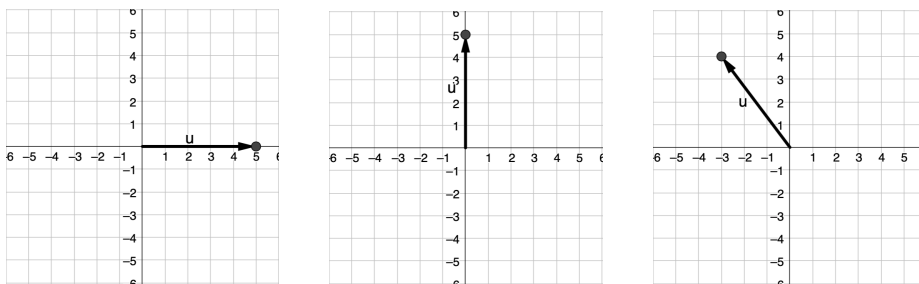
Exercise 2

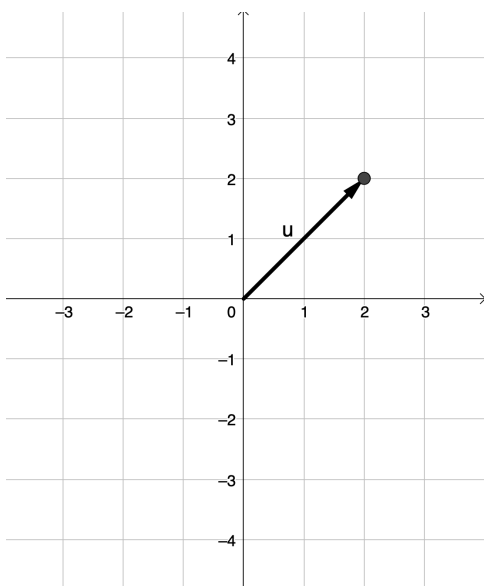
Complete the following vector operations

- $(3, 4) + (4, 3) =$
- $(2, 2) - (1, 2) =$
- $((2, 1) + (7, 3)) - (1, 6) =$
- $(\frac{1}{2}, \frac{3}{4}) + (\frac{1}{2}, \frac{1}{4}) =$

Magnitudes

Although the vectors tell us where the force is going, we don't know yet how strong the force actually is! For that, we take the magnitude of the vector. The equation for the magnitude of the vector comes from one of the most well known, and famous theorems in all of existence, the Pythagorean theorem. We'll construct it from the ground up. Recall that the magnitude of a vector is the *length* of the vector. So, as I quickly mentioned in the beginning of the lesson, the following vectors have magnitude 5, but this is easily seen as they are just along the axis.





But what if we wanted to find the magnitude of this vector? How would we calculate the magnitude of a vector which is on a diagonal? There's a few things we have to cover first

- Square: Squaring a number means to multiply the number by itself.
- Square roots: The opposite of square a number, taking the square root of a number, n , returns the number that you have to square to get n back.

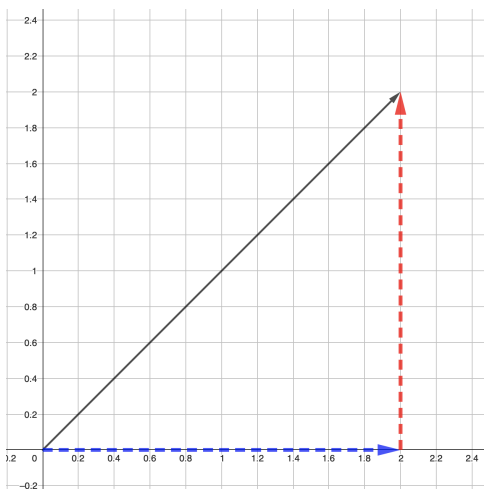
Squaring a number looks like $2^2 = 2 \cdot 2$, and taking the square root looks like $\sqrt{2}$. Very often, when taking the square root, you get a number with many decimal places, so that's why you can use your calculator, by pressing the $\sqrt{\quad}$ button.

Exercise 3

Compute the following expressions by hand or on your calculator.

$$2^2 \quad 7^2 \quad \sqrt{25} \quad \sqrt{8} \quad \sqrt{(4^2)} \quad (\sqrt{2})^2 \quad (-2)^2$$

That being said, when we have any vector, we can decompose it into their x and y component. We'll use blue and red to show the x and y component respectively. Notice that when we add the red and blue vector together (the x and y component), we get the original vector back, so in practice we can work with the decomposed vectors and it's the same as the original vector.



Again, we wish to determine the length (magnitude) of the black vector shown on the right. Pythagoras' theorem states that

$$(x - \text{component})^2 + (y - \text{component})^2 = (\text{length})^2$$

So, using our notions of "squaring" and "square roots", we can solve for the length

$$\begin{aligned} \sqrt{(\text{length})^2} &= \sqrt{(x - \text{component})^2 + (y - \text{component})^2} \\ (\text{length}) &= \sqrt{(x - \text{component})^2 + (y - \text{component})^2} \end{aligned}$$



Now we have the magnitude of the diagonal vector. Notice that when the x and y components meet tip to tail, they form a right angle (90°). Pythagoras' theorem, stated mathematically is

Pythagorean theorem

Given a right angle triangle with side lengths a and b , with hypotenuse c . we have that

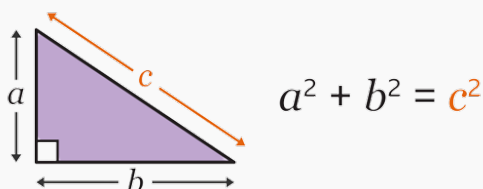


Image retrieved from [BBC](#)

Note that the hypotenuse is the side length opposing the right angle.

Example 1

Using the x and y components of the vectors above in **Exercise 1**, determine the magnitude of each vector.

Solution: In the first image, we have that $x = 1$ and $y = 1$, thus we have

$$\text{magnitude} = \sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2} \approx 1.41421356237\dots$$

In the second, we have that $x = -1$ and $y = 2$

$$\text{magnitude} = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5} \approx 2.2360679775\dots$$

In the third, we have that $x = 3$ and $y = -2$

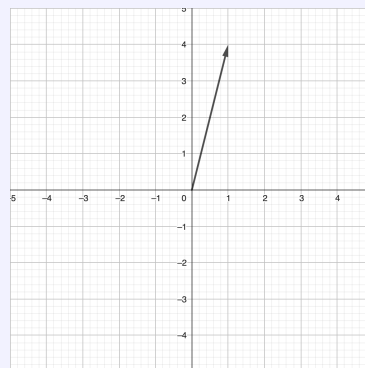
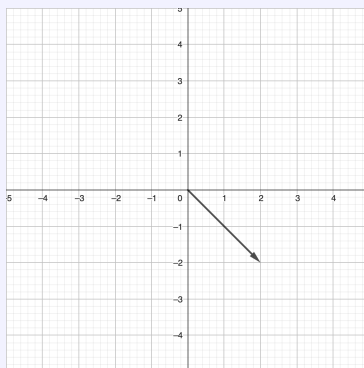
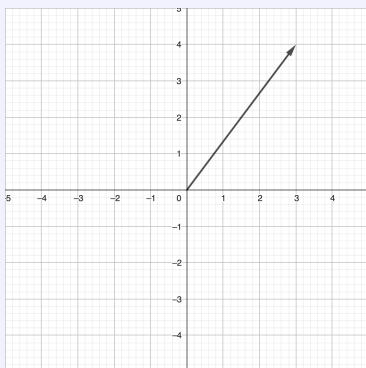
$$\text{magnitude} = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13} \approx 3.60555127546\dots$$

One thing to mention is that we denote the magnitude of a vector, \vec{v} for example, as $\|\vec{v}\|$. So, the magnitude of (a, b) is denoted as $\|(a, b)\| = \dots$



Exercise 4

Determine the magnitude of the following vectors



Back to Physics!

Before moving on, let's look back at the law and define some terms

Definitions

- Net force: the net force is the vector sum of all the forces acting on an object
- Acceleration: acceleration is the rate of change of the velocity of an object with respect to time (i.e how much the velocity is changing per second)
- Mass: This one is difficult to define! For our lesson, it's how "heavy" an object is. It is the measure of how hard it is to get an object moving from rest

This law relates the total force acting on an object to its acceleration, mathematically,

$$\vec{F}_{\text{tot}} = m\vec{a}$$

From this, we want to find the total **net force** acting on an object, so when we have forces applied to a body (i.e vectors), we wish to add them, and see where the resulting force is. For example



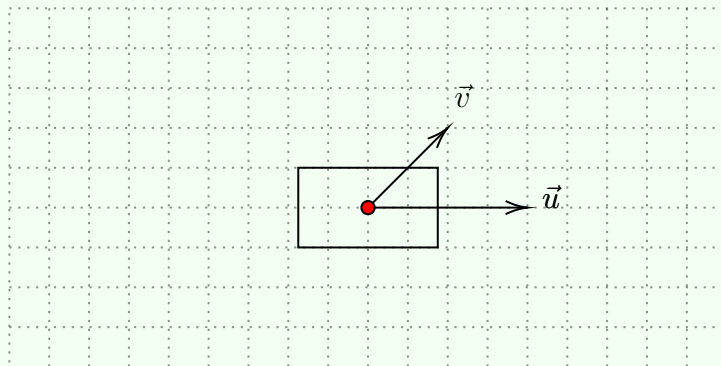
In the image above, we have a mysterious force pushing to the right, and a force pushing to the left,



so the **net force** are these two forces added together. Which direction do you think the box will move? Which force is stronger, and will dominate? Let's actually go through an example together.

Example 2

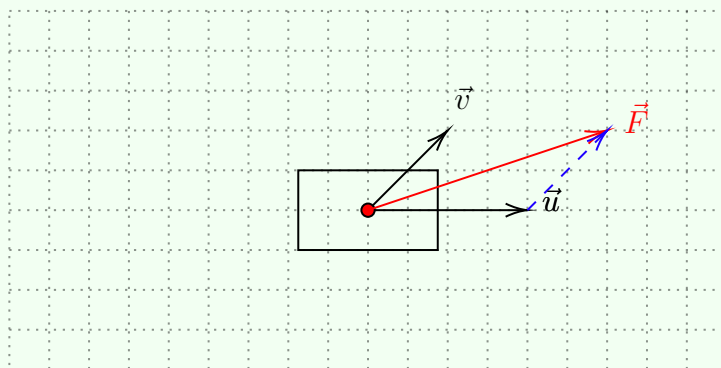
Given the following diagram, determine the net force acting on the box, and draw the resulting vector.



Solution: For artistic brevity, I've denoted $(0, 0)$, the origin, with a red dot. From it, you can still find the vector form for each vector with ease by counting the boxes. In our case, we have that $\vec{v} = (2, 2)$, and that $\vec{u} = (4, 0)$. Adding these vectors (to get the vector representing the **net force**), we get

$$\vec{F} = \vec{v} + \vec{u} = (2, 2) + (4, 0) = (6, 2)$$

Visually, the resulting vector looks like



With this, the net force is therefore the magnitude of the vector \vec{F}

$$\|\vec{F}\| = \sqrt{6^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40} = 6.32\dots$$

**Stop and Think**

Can the magnitude of the vector ever be a negative number? What would that say about forces acting on objects?

The magnitudes also represent how much force is behind an action and we quantify this magnitude with the unit called *Newtons*. Since we said that $F = ma$, and we know that m has units of kg , and a has units of m/s^2 , we have that the force carries units of $m \cdot kg/s^2$. This is called a Newton. Thus the magnitude represents how many Newton's of force is being applied in the specific direction. In Example 2, that means there was a total force of approximately 6.32 Newtons applied up and to the right if we take one unit distance on the grid to represent one Newton of force.

Example 3

Using the same force in Example 2, if the box weighed $10kg$, determine the acceleration of the box.

Solution: We know by Newton's second law that

$$F = ma$$

Rearranging for a by dividing by m , we get that

$$\frac{F}{m} = \frac{ma}{m} = a$$

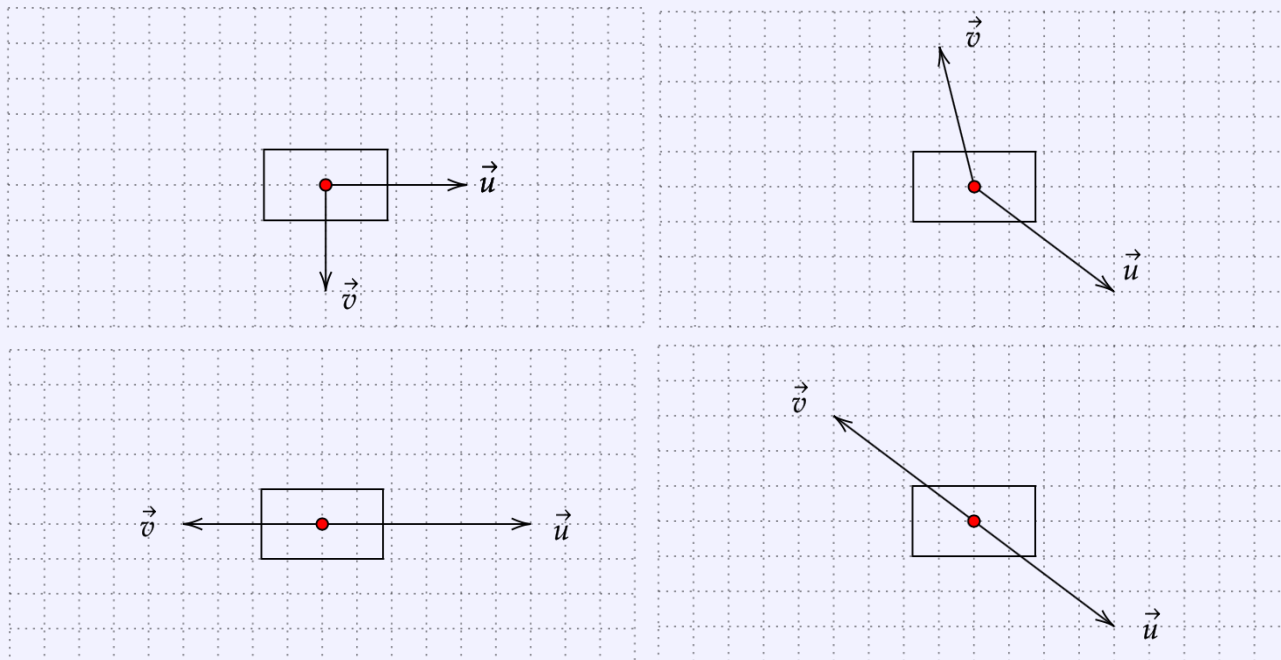
Since we found that $F = 6.32N$, and we're told that $m = 10kg$, we have that

$$a = \frac{F}{m} = \frac{6.32N}{10kg} = 0.632m/s^2$$

Newton's second laws are extremely important for understanding basic kinematics, the study of moving objects. With a simple diagram, we as physicists are able to determine where the object will be going, how fast it is moving or accelerating, or how heavy the object is. Fundamentally, that's all you need to know about the object. With that, you can predict where it is, where it was, and where it will be. since Newton, we've come a far way, and we've advanced our notion of moving bodies, on the very very small scale, to the very large.

Exercise 5

Determine the net force acting on each object, and determine the acceleration if each box weighs $5kg$. Take each unit on the grid to represent $1N$ of force



We've now seen a very important relationship between vectors and physics! Physicists like using vectors to describe their work, as the idea that a mathematical object can contain direction is very important for when things interact with each other and move around. This lesson didn't explore in detail the direction of our net force, as that requires some more mathematical background. In particular, to understand directions of vectors in non-trivial positions, you need to understand trigonometry. Trigonometry is the branch of math which investigates the relationship between the ratio of side lengths of triangles to interior angles. Newton's second law is a great example of how physical realities are explained by mathematical formula. We cannot prove these laws, but after more than 300 years, they have yet to fail in describing real world phenomenon.